Practice 2nd Exam Solutions

1.) Know $(0,1)$ is uncountable. Prove $(1,\infty)$ is uncountable.

pf:
Consider the func $f(x)=e^{x}:(0,1)\to(1,\infty)$ then $f(0)=e^{0}=1$ and $\lim_{x\to\infty} e^{x}=\infty$
so given any $y\in(1,\infty)$ $e^{x}=y \Rightarrow x=\ln y\in(0,1)$. so $f(x)$ is onto. next
since $x_1,x_2\in(0,1)$ consider $e^{x_1}=e^{x_2}$ then $\ln(e^{x_1})=\ln(e^{x_2}) \Rightarrow x_1=x_2$
so f is 1-1. thus $(0,1)$ is bijective w/ $(1,\infty)$ since $(0,1)$ uncountable $\Rightarrow (1,\infty)$ uncountable. □

2.) Show $\mathbb{Z}$ is countable by constructing an explicit map from $\mathbb{N}$ to $\mathbb{Z}$.

pf:
define $f:\mathbb{N}\to\mathbb{Z}$ as follows $f(n)=\begin{cases}\frac{n}{2} & \text{if } n \text{ even}\\ -\frac{n-1}{2} & \text{if } n \text{ odd}\end{cases}$ 1st consider injective
there are 2 cases. (1) let n_1,n_2 be even suppose $f(n_1)=f(n_2) \Rightarrow \frac{n_1}{2}=\frac{n_2}{2} \Rightarrow n_1=n_2$
(2) let n_1,n_2 be odd then since $f(n_1)=f(n_2) \Rightarrow -\frac{n_1-1}{2}=-\frac{n_2-1}{2} \Rightarrow n_1-1=n_2-1 \Rightarrow n_1=n_2$
∴ f is injective. Next for surjective there are 2 cases also. since $m\in\mathbb{Z}$.
(1) if $m\geq 0$ then $f(n)=m \Rightarrow \frac{n}{2}=m \Rightarrow n=2m$ or n even. (2) if $m<0$
then $f(n)=m \Rightarrow -\frac{(n-1)}{2}=m \Rightarrow n=-2m+1$ & remember $m<0$ so $-m>0$
so $-2m+1$ is odd. so can always find n from domain. Thus f is surjective
so $f:\mathbb{N}\to\mathbb{Z}$ bijective hence $\mathbb{Z}$ countable. as $\mathbb{N}$ is. □

3.) let $\alpha\in\mathbb{R}$ and α is transcendental. let $p(x)\in\mathbb{Z}[x]$, Prove $p(\alpha)$ is transcendental.

pf:
Suppose $p(\alpha)$ is not transcendental thus $p(\alpha)$ is algebraic, so $\exists$ poly $q(x)\in\mathbb{Z}[x]$
s.t. $q(p(\alpha))=0$ but $(q\circ p)(x)\in\mathbb{Z}[x]$ so $(q\circ p)(\alpha)=q(p(\alpha))=0 \Rightarrow \alpha$ is algebraic
w/ poly $q\circ p$ contradiction as α is transcendental. □

4.) let $\alpha\in\mathbb{R}$ be algebraic. Express α^{-1} in terms of $d_k\alpha^k$ for $d_k\in\mathbb{Q}$

pf:
Since α algebraic, $\exists$ poly $p(x)\in\mathbb{Z}[x]$ s.t. $p(\alpha)=0$. let $p(x)=c_0+c_1x+\dots+c_nx^n$
then $c_n\alpha^n+c_{n-1}\alpha^{n-1}+\dots+c_0=0 \Rightarrow c_n\alpha^n+\dots c_1\alpha=-c_0 \Rightarrow \alpha(c_n\alpha^{n-1}+\dots+c_1)=-c_0$
$\Rightarrow \alpha^{-1}=-\frac{1}{c_0}(c_n\alpha^{n-1}+\dots+c_1)=\frac{-c_n}{c_0}\alpha^{n-1}-\frac{c_{n-1}}{c_0}\alpha^{n-2}-\dots-\frac{c_1}{c_0}$
thus clearly $\frac{c_j}{c_0}\in\mathbb{Q}$ for $j=1,\dots,n$ □

5.) Prove $5n+3$ and $7n+4$ are relatively prime for all $n\in\mathbb{N}$.

pf:
recall $\gcd(a,b)=1 \Rightarrow a,b$ relatively prime and $\gcd(a,b)=as+bt$ for some $s,t\in\mathbb{Z}$.
Consider $7(5n+3)-5(7n+4)=35n+21-35n-20=21-20=1$
so $\gcd(5n+3,7n+4)=1$ □

6.) let $m, n \in \mathbb{Z}$ s.t. $\gcd(m,n) = 1$. Prove $\forall r \in \mathbb{Z}$ $\exists x, y \in \mathbb{Z}$ s.t. $r = mx + ny$.

pf:

since $\gcd(m,n) = 1$ and we know $\gcd(m,n) = sm + tn$ for some $s, t \in \mathbb{Z}$ we have $ms + nt = 1$ then let $r \in \mathbb{Z}$ arbitrary $\Rightarrow r = msr + ntr$ s.t. $x = sr$ and $y = tr$. Since $s, t \in \mathbb{Z}$ and $r \in \mathbb{Z}$ then $sr, st \in \mathbb{Z}$ so $x, y \in \mathbb{Z}$. □

7.) let $d = \gcd(a,b)$ w/ $a, b \in \mathbb{N}$. If $a = db'$ and $b = db'$ show $\gcd(a', b') = 1$.

pf:

know $\gcd(a,b) = sa + tb$ for some $s, t \in \mathbb{Z}$. OTOH since $d = \gcd(a,b)$ get $d = as + bt$ but $a = da'$ and $b = db'$ $\Rightarrow$ $d = da's + db't$ .. dividing by d yields $1 = a's + b't$. $d \neq 0$ unless a or $b = 0$. but $a, b \in \mathbb{N}$. □.

8.) let $d = \gcd(a,b)$ w/ $a, b \in \mathbb{N}$. prove $\frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

pf:

want to show $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$. know $\gcd(a,b) = sa + tb$ for some $s, t \in \mathbb{Z}$ so like #7 $d = as + bt$ $\Rightarrow$ $1 = \frac{a}{d}s + \frac{b}{d}t$ or $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$. □.

9.) Prove $(0,1)$ is uncountable by constructing a bijection from $(0,1)$ to $\mathbb{R}$.

pf:

let $f(x) = \tan(\pi x - \frac{\pi}{2}) : (0,1) \longrightarrow \mathbb{R}$. notice $f(x) = (g \circ h)(x)$ w/ $g(x) = \tan x$ and $h(x) = \pi x - \frac{\pi}{2}$ where $h(x) : (0,1) \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ and $g(x) : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ just need to demonstrate g, h are bijective as composition of bijections is bijective. We proved before that $\tan x$ is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. so $g(x)$ is. Just need to show $h(x)$ is. let $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ then $h(x) = y \Rightarrow y = \pi x - \frac{\pi}{2}$ or $x = \frac{1}{\pi}(y + \frac{\pi}{2}) \in (0,1)$ if $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. so $h(x)$ is surjective. next if $x_1, x_2 \in (0,1)$ consider $h(x_1) = h(x_2) \Rightarrow \pi x_1 - \frac{\pi}{2} = \pi x_2 - \frac{\pi}{2} \Rightarrow \pi x_1 = \pi x_2 \Rightarrow x_1 = x_2$ so h is 1-1. Thus h is a bijection. Thus $f(x)$ is a bijection between $(0,1)$ and $\mathbb{R}$ and since $\mathbb{R}$ uncountable $\Rightarrow$ $(0,1)$ uncountable. □